5 Examples

Do as many as you can!

Problem 1. Use the table of values of $f(x, y)$ to estimate the values of $f_x(3, 2)$ and $f_y(3, 2)$.

$x \backslash y$	1.8	2.0	2.2
2.5	12.5	10.2	9.3
3.0	18.1	17.5	15.9
3.5	20.0	22.4	26.1

$$f_x(3,2) = \lim_{h\to 0} \frac{f(3+h,2) - f(3,2)}{h}$$

$$h = 0.5: \quad \frac{f(3.5,2) - f(3,2)}{0.5} = 9.8$$

$$h = -0.5: \quad \frac{f(2.5,2) - f(3,2)}{-0.5} = 14.6$$

$$\Rightarrow f_x(3,2) \approx \frac{9.8 + 14.6}{2} = 12.2$$

$$f_y(3,2) = \lim_{h\to 0} \frac{f(3,2+h) - f(3,2)}{h}$$

$$h = 0.2: \quad \frac{f(3,2.2) - f(3,2)}{0.2} = -8$$

$$h = -0.2: \quad \frac{f(3,1.8) - f(3,2)}{-0.2} = -3$$

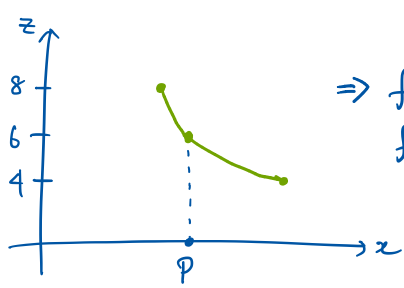
$$\Rightarrow f_y(3,2) \approx \frac{-8-3}{2} = -5.5$$

Problem 2. Consider the level curves given in Example 2. Determine whether the following partial derivatives are positive or negative at the point P.

a. $f_{xx} > 0$

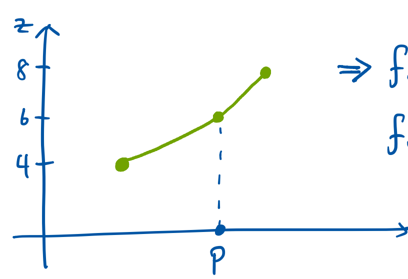
b. $f_{yy} > 0$

c. $f_{xy} < 0$



$\Rightarrow f_x < 0$

$f_{xx} > 0$



$\Rightarrow f_y > 0$

$f_{yy} > 0$

f_{xy} : how does f_x change as y increases?

- contours farther apart in x-direction below P $\Rightarrow$ f_x is less negative
- contours closer together in x-direction above P $\Rightarrow$ f_x is more negative

$\Rightarrow f_{xy} < 0$

Problem 3. Let $f(x,y) = \arctan(\overbrace{y/x}^{yx^{-1}})$. Find $f_x(2,3)$.

$$\frac{d}{du}\underbrace{\arctan}_{\tan^{-1}}(u) = \frac{1}{1+u^2}$$

$$f_x(x,y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-yx^{-2}\right) = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\Rightarrow f_x(2,3) = -\frac{3}{2^2+3^2} = -\frac{3}{13}$$

Problem 4. Let $f(x,y,z) = \frac{\overbrace{y}^{y(x+y+z)^{-1}}}{x+y+z}$. Find $f_y(2,1,-1)$.

$$\frac{u}{v} \rightarrow \frac{vu' - uv'}{v^2}$$

(Partial derivatives of functions of 3 or more variables are found the same way: regard all but one variable as constant, and take the derivative with respect to the remaining variable.)

$$f_y(x,y,z) = \frac{(x+y+z)(1) - (y)(1)}{(x+y+z)^2} = \frac{x+z}{(x+y+z)^2}$$

$$\Rightarrow f_y(2,1,-1) = \frac{2-1}{(2+1-1)^2} = \frac{1}{4}$$

Problem 5. Let $f(x,y,z) = \sqrt{\sin^2 x + \sin^2 y + \sin^2 z}$. Find $f_x(0,0,\pi/4)$.

$$f(x,y,z) = \left(\sin^2 x + \sin^2 y + \sin^2 z\right)^{\frac{1}{2}}$$

$$f_x(x,y,z) = \frac{1}{2}\left(\sin^2 x + \sin^2 y + \sin^2 z\right)^{-\frac{1}{2}}\left(2\sin x \cos x\right)$$

$$= \frac{\sin x \cos x}{\sqrt{\sin^2 x + \sin^2 y + \sin^2 z}} \qquad \Rightarrow f_x\left(0,0,\frac{\pi}{4}\right) = 0$$

Problem 6. Find all the second partial derivatives of $f(x,y) = \overbrace{x^4y - 2x^3y^2}^{x^4y - 2x^3y^2}$.

$$f_x(x,y) = 4x^3y - 6x^2y^2 \qquad f_y(x,y) = x^4 - 4x^3y$$

$$f_{xx}(x,y) = 12x^2y - 12xy^2 \qquad f_{yy}(x,y) = -4x^3$$

$$f_{xy}(x,y) = 4x^3 - 12x^2y \qquad f_{yx}(x,y) = 4x^3 - 12x^2y$$

Problem 7. Let $f(x, y) = \cos(x^2y)$. Verify that Clairaut's theorem holds: $f_{xy} = f_{yx}$.

$$f_x(x,y) = -\underbrace{\sin(x^2y)}_{u}\underbrace{(2xy)}_{v} \qquad f_y(x,y) = -\underbrace{\sin(x^2y)}_{u}\underbrace{(x^2)}_{v}$$

$$f_{xy}(x,y) = \underset{uv'}{-\sin(x^2y)(2x)} + \underset{vu'}{(2xy)\big(-\cos(x^2y)(x^2)\big)} = -2x\sin(x^2y) - 2x^3y\cos(x^2y)$$

$$f_{yx}(x,y) = \underset{uv'}{-\sin(x^2y)(2x)} + \underset{vu'}{(x^2)\big(-\cos(x^2y)(2xy)\big)} = -2x\sin(x^2y) - 2x^3y\cos(x^2y)$$

Problem 8. Let $f(x, y) = \sin(2x + 5y)$. Find f_{yxy}.

$$f_y(x,y) = \cos(2x+5y)(5) = 5\cos(2x+5y)$$

$$f_{yx}(x,y) = \big(-5\sin(2x+5y)\big)(2) = -10\sin(2x+5y)$$

$$f_{yxy}(x,y) = \big(-10\cos(2x+5y)\big)(5) = -50\cos(2x+5y)$$

Problem 9. Find all the second partial derivatives of $f(x, y) = \ln(ax + by)$.

$$f_x(x,y) = \frac{1}{ax+by}(a) = a(ax+by)^{-1} \qquad f_y(x,y) = \frac{1}{ax+by}(b) = b(ax+by)^{-1}$$

$$f_{xx}(x,y) = -a(ax+by)^{-2}(a) = -a^2(ax+by)^{-2}$$

$$f_{yy}(x,y) = -b(ax+by)^{-2}(b) = -b^2(ax+by)^{-2}$$

$$f_{xy}(x,y) = -a(ax+by)^{-2}(b) = -ab(ax+by)^{-2}$$

$$f_{yx}(x,y) = f_{xy}(x,y)$$